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On the Collision between two PNG Droplets

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In this article we study the interface generated by the collision between two crystals growing layer by layer on a one-dimensional substrate through random decomposition of particles. We relate this interface with the notion of β-path in an equivalent directed polymer model and, by using asymptotics results from J. Baik and E. Rains, J. Stat. Phys., **100**:523–541 (2000). and some hydrodynamic tools introduced by E. Cator and P. Groeneboom, Ann. Probab., **33**:879–903 (2005), we derive a law of large numbers for such a path and obtain some bounds for its fluctuations.

KEY WORDS: polynuclear growth model, directed polymer model, Hammersley's process with sinks and sources, second-class particle, competition interface, hydrodynamic equation, characteristics

2000 Mathematics Subject Classification: 60C05, 60K35

1. INTRODUCTION

A variety of one dimensional growth models has been proposed to understand the interplay between the geometry of the initial macroscopic profile and the scaling properties of the growing interface.[(18)] A less well understood phenomenon is the interface generated by the collision between two growing materials, named the *competition interface.*[3] Since the numerical simulations performed by Derrida and Dickman[(8)] it is well known that the large space and time behavior of this interface strongly depends on the geometry of the initial profile (see also Ref. 24). Later, Ferrari, Martin and Pimentel[(12)] considered the competition interface between two clusters in the lattice last-passage percolation set-up and they established a connection between this interface and the so called second-class particle in the

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[3] We follow the terminology introduced by Ferrari and Pimentel.[(11)]

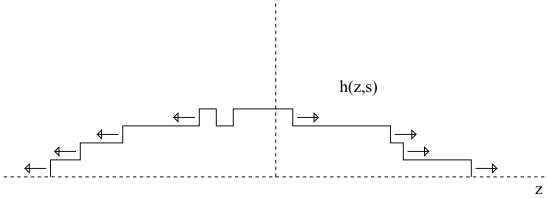

Fig. 1. Single PNG droplet.

totally asymmetric exclusion process. This connection allowed them to perform formal calculations and obtain analytical solutions for the macroscopic description of the competition interface.

In this work we do something similar but now in the context of a one dimensional layer by layer growth model,[(19)] named the *polynuclear growth* (PNG) model. This model describes a crystal growing layer by layer on a one dimensional substrate through random deposition of particles that nucleate on the existing plateaus of the crystal forming new islands. These islands spread laterally with speed 1 and adjacent islands of the same level coalesce upon meeting (Fig. 1).

To consider a multi-type crystal growth model[4] we assume that the initial substrate is divided into two different types of crystals, say type 1 if $z < 0$ and type 2 if $z > 0$. The dynamics stipulate that if a nucleation occurs on an existing plateau of type $j \in \{1, 2\}$ then the new island will be of the same type. When edges of islands having different types meet, they stop (Fig. 2).

Of course the behavior of this model depends on the geometry of the nucleation events, which can be seen as a point process $\mathcal{N}$ in (z, s)-space-time. We restrict our attention to a particular class of point processes $\mathcal{N}$ (PNG growth with external sources) and we show a law of large numbers for the competition interface.

The PNG model can be studied in a directed polymer context[(22)] and we show that the space-time path of the competition interface is a particular example of a β-*path*. The collection of β-paths form a large class of paths in the directed polymer model and we prove a general theorem that ensures its almost sure convergence. We also show that the exponent whose value measures the order of the fluctuations of a β-path about its asymptotic value is at most $2/3$. The proofs are based on the notion of maximal paths and its relation with β-paths, together with some bounds for the tail of the length of the longest directed polymer connecting two distinct points on the space-time plane obtained by Baik and Rains.[(3)]

The PNG and the directed polymer models are intrinsically related to the Hammersley's interacting particle system and another example of a β-path arises naturally in this context, namely the second-class particle. As a consequence, we

[4] For more information on multi-type growth models we address to Ref. 21 and the references therein.

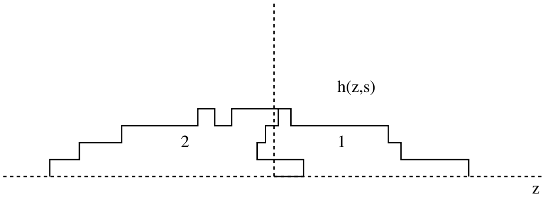

Fig. 2. Two PNG droplets.

obtain a law of large numbers for this special particle which, together with some hydrodynamic ideas introduced by Cator and Groeneboom,[5] will also play and important role in studying the asymptotics of β-paths.

In the next sections we formally introduce all the models considered here and we state the main results.

1.1. Polynuclear Growth with External Sources

The surface at time $s \geq 0$ is described by an integer-valued function $h(.,s) : z \in \mathbb{R} \to h(z,s) \in \mathbb{Z}$, named the height profile at time s, for which the discontinuity points have upper limits. We consider the initial condition $h(0,z) = 0$ for all $z \in \mathbb{R}$. For each $s > 0$ the function $h\,(\,.,s)$ has jumps of size one at the discontinuity points, called *up-step* if h increases and *down-step* if h decreases (Fig. 1). A *nucleation* event at position (z,s) is a creation of a spike, a pair of up- and down-steps, over the previous layer. The up-steps move to the left with unit speed and the down-steps move to the right with unit speed. When an up- and down-step collide, they disappear.

The nucleation events form a locally finite point process in space-time. On $\{z = s\}$ we put a Poisson point process $\mathcal{D}_+$ of intensity $\lambda \geq 0$ while on $\{z = -s\}$ we put a Poisson point process $\mathcal{D}_-$ of intensity $\rho \geq 0$. On $\{|z| < s\}$ we put a Poisson point process $\mathcal{D}$ of intensity 1. We assume that outside $\{|z| \geq s\}$ there is no nucleation event and that all the Poisson point processes involved in this construction are mutually independent. To know the value of $h(z,s)$ one draws the trajectories of the up- and down-steps in the space-time (z,s)-plane. When two of these paths meet (as s increases) they stop, which reflects the disappearing of the corresponding up- and down-step. In this way the space-time is divided into regions bounded by piecewise straight lines with slopes equal to 1 or -1 (Fig. 3). For fixed $z \in \mathbb{R}^2$ the height $h(z,s)$ is constant in each region.

To introduce the multi-type growth model we assume that the initial layer is divided into two different types of crystals, say 1 if $z < 0$ and 2 if $z > 0$. Consider the rule which stipulates that if a spike is created over crystal j then it will belong to this material, and that when a down-step of type 1 collides with an up-step of type 2 they stop (Fig. 2). Thus an interface between these two growing crystals

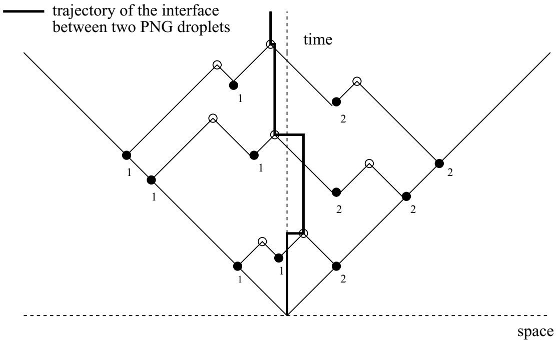

Fig. 3. The nucleation events. Each point is labeled by its type.

is formed. We denote by φ_n the position in the z-axis of the collision between the different types of crystals at the n-th layer and by σ_n the time for which this happens, with the convention that $\varphi_0 = 0$ and $\sigma_0 = 0$. Define the process $(\varphi(s),\ s \geq 0)$ by setting $\varphi(s) = \varphi_n$ for $s \in [\sigma_n, \sigma_{n+1})$. We prove:

Theorem 1. *Assume that* $0 \leq \lambda\rho \leq 1$. *One has almost surely that*

$$\lim_{s \to \infty} \frac{\varphi_s}{s} = W\,.$$

where

$$W \in \left[\frac{\rho^2 - 1}{\rho^2 + 1}, \frac{1 - \lambda^2}{1 + \lambda^2} \right].$$

Remark 1. In the stationary regime $\lambda\rho = 1$ we obtain a deterministic limiting value for the inclination of the competition interface. Namely, $W = \frac{1-\lambda^2}{1+\lambda^2}$. In Sec. 2 we discuss the correspondence between the limiting value W and the macroscopic behavior of the height profile.

1.2. Directed Polymer Set-up and β-paths

There is a link between the PNG growth model and a model for directed polymers on Poisson points.[23] This directed polymer model can be regarded as a last-passage percolation model on $\{x \geq 0\,,\ t \geq 0\}$ and is defined as follows. Put a Poisson point process $\mathcal{P}$ of intensity 1 in the strictly positive quadrant $\{x > 0\,,\ t > 0\}$. Independently of $\mathcal{P}$ we also have mutually independent Poisson point processes, say $\mathcal{X}$ and $\mathcal{T}$, on the x- and t-axis and of intensities $\lambda, \rho \geq 0$, respectively. For P and Q on the plane, define that $P \prec Q$ if both coordinates of P are lower or equal than those of Q. For a given realization of the three Poisson point processes, a weakly up/right path, or directed polymer, $(P, P_1, \dots, P_l, Q)$, starting at P and ending at Q, is an oriented and piecewise linear path γ connecting $P \prec P_1 \prec \cdots \prec P_l \prec Q$, where each P_j is a Poissonian point for $j = 1, \dots, l$. The length $l(\gamma)$ of the path is the number of Poissonian points used by γ and $\Gamma(P, Q)$ denotes the set of all weakly up/right paths from P to Q. The maximal length, or the last-passage time, between P and Q is defined by

$$L_m(P, Q) = \max_{\gamma \in \Gamma(P,Q)} l(\gamma) \ . \tag{1.1}$$

Every $\gamma \in \Gamma(P, Q)$ such that $l(\gamma) = L_m(Q)$ is called a maximal path. We as lo consider the growth process $(G_k)_{k \geq 0}$ defined by

$$G_k = \{Q \ : \ L_m(0, Q) \leq k - 1\}$$

for $k \geq 1$ and for convenience we set $G_0 := \{(0, 0)\}$. We denote by ∂G_k the right (hand-side) boundary of G_k.

Consider the transformation $A \ : \ (x, t) \to (z, s)$ that rotates the (x, t)-plane by 45° in the anti-clockwise orientation. If the Poisson point processes involved in the construction of both processes are related by A, $A(\mathcal{X}) = \mathcal{D}_-$, $A(\mathcal{T}) = \mathcal{D}_+$ and $A(\mathcal{P}) = \mathcal{D}$, then the link is apparent. In fact, $h(z, s)$ equals the number of lines crossed by any piecewise linear path from (0, 0) to (z, s), with slope between -1 and 1. In particular, one considers the paths which cross them at the nucleation points. These are maximal paths introduced above up to a 45° rotation, and thus it follows that $h(z, s) = L_m(0, (x, t))$.

To see the rule, in this directed polymer model, played by the competition interface we introduce the notion of *β-paths*. A *β-point*[5] $Q \in \mathbb{R}^2_+$ is a concave corner of ∂G_k for some $k \geq 1$. Thus, in the PNG model, β-points will corresponds to the collisions between up- and down- steps, up to a 45° rotation. We define that $(P_n)_{n \geq 0}$, a sequence of points in $\{x \geq 0 \, , \, t \geq 0\}$, is a β-path if it satisfies: (i) $P_n \prec P_{n+1}$; (ii) $P_n \in \partial G_n$; (iii) P_n is a β-point. Recalling that φ_n denotes the position in the z-axis of the collision between the different types of crystals at the n-th layer, and that σ_n denotes the time in which this happens, one can see that the path $(R_n)_{n \geq 0}$, where $R_n := (\varphi_n, \sigma_n)$, is a β-path up to a 45° rotation (see the trajectory in Fig. 3).

For $P = |P|(\cos\theta, \sin\theta)$ and $Q = |Q|(\cos\alpha, \sin\alpha)$, with $\alpha, \theta \in [0, \pi/2]$, let $ang(P, Q) = |\beta - \alpha|$ be the angle in $[0, \pi/2)$ between P and Q. We prove:

[5] We follow the terminology introduced by Groeneboom[(14)] in the Hammersley's process context (see Sec. 1.3).

Theorem 2. *Assume that* $0 \le \lambda\rho \le 1$. *One has almost surely that, if* $(P_n)_{n\ge1}$ *is a* β*-path, then*

$$\exists \lim_{n\to\infty} \frac{P_n}{|P_n|} = V = (\cos\theta, \sin\theta),$$

where $tan(\theta) \in [\lambda^2, \rho^{-2}]$.

We remark that Theorem 1 follows directly from Theorem 2. Concerning the fluctuations around its asymptotic angle we have:

Theorem 3. *Assume that* $0 \le \lambda\rho < 1$. *Then for all* $\delta \in (0, 1/3)$ *there exists a constant* $c > 0$ *such that, almost surely,*

$$ang(P_m, V) \le c|P_m|^{-\delta} \textit{ for all large } m\,.$$

We note that Theorem 3 tell us that, in the regime $0 \le \lambda\rho < 1$, for all $\epsilon > 0$ the fluctuations of a β-path $(P_n)_{n\ge0}$ about its asymptotic value $V|P_n|$ are at most of order $|P_n|^{2/3+\epsilon}$. We do believe that Theorem 2 is almost optimal, i.e. that the correct exponent should be $2/3$.

1.3. Hammersley's Process and Second-class Particles

Aldous and Diaconis[1] introduced a continuous time version of the interacting particle process in Hammersley[15] using the following rule. Start with the Poisson point process $\mathcal{P}$ on $\{x > 0\,,\ t > 0\}$, of intensity 1, and move the interval $[0, x]$ vertically through a realization of this point process; if this interval catches a point that is to the right of the points caught before, a new point (or particle) is created in $[0, x]$ at this point; otherwise we shift to this point the previously caught point that is immediately to the right and belongs to $[0, x]$. The number of particles, resulting from this rule, at time t on the the interval $[0, x]$ is denoted by $N(x, t)$ and the evolving particle process $(N(., t)\,, t \ge 0)$ is called the *Hammersley's process*. In this work we consider an extension of the Hammersley's process, as introduced by Groeneboom,[14] where we also have two others Poisson point processes $\mathcal{X}$ and $\mathcal{T}$, of intensities λ and ρ and on the x- and t-axis, respectively. Points in $\mathcal{X}$ are called sources while points in $\mathcal{T}$ are called sinks. Now we have the following rule: start the interacting particle process with a configuration of sources on the x-axis, which are subjected to the Hammersley interacting rule in the strictly positive quadrant and which escape through the sinks on the t-axis, if such a sink appears to the immediate left of a particle. Now, $N(x, t)$ is the number of particles in $(0, x] \times \{t\}$ plus the number of sinks in $\{0\} \times [0, t]$. When $\rho = 1/\lambda$, we have a stationary process.[14]

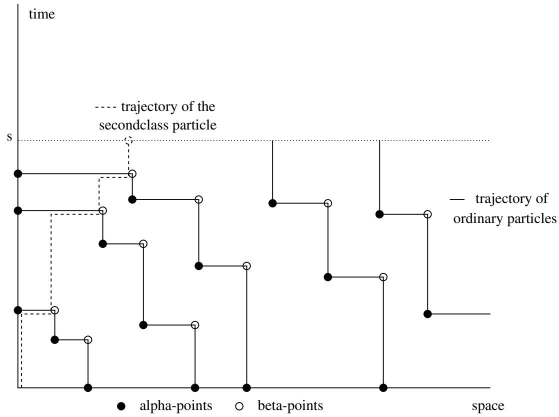

Fig. 4. Second-class particle

Denote by $\Delta_0, \Delta_1, \Delta_2 \ldots$ the space-time paths of the Hammersley particles with the convention that $\Delta_0 = \{(0, 0)\}$ and that Δ_k lies below Δ_{k+1}. Thus, ∂G_k equals Δ_k (recall we have constructed both process with the same Poissonian points). Again, if the Poissonian processes are related by A, the rotated space-time paths of the up- and down steps correspond to the space-time paths of Hammersley particles. With this picture in mind, one can also see that the β-points are the left turns of the space-time paths of the particles in the Hammersley's process (Fig. 4). We remark also that, in the stationary regime $\lambda\rho = 1$, Cator and Groeneboom[5] proved that the β-points inherit the Poisson property of $\mathcal{P}$, which allows us to see a duality between β-paths and maximal paths: a finite β-path is a maximal path for the time reversal process.

It turns out that another example of a β-path appears naturally in the Hammersley's process: the so called *second-class particles*. A normal second-class particle is a special particle that starts at the origin and jumps to the previous position of the ordinary Hammersley particle that exits through the first sink at the time of the exit, and successively jumps to the previous position of particles directly to the right of it, at times where these particles jump to a position to the left of the second-class particle (Fig. 4). The position of the second-class particle at time t is denoted by X_t. Thus, if τ_n denotes the time of the n-th jump of the second-class particle (with the convention that $\tau_0 = 0$) then $(Q_n)_{n\geq 0}$, where $Q_n := (X_{\tau_n}, \tau_n)$, is a β-path.

Remark 2. In the stationary regime $\lambda\rho = 1$ Cator and Groeneboom[5] proved that, almost surely,

$$\lim_{t\to\infty} \frac{X_t}{t} = \frac{1}{\lambda^2}\,.$$

They also showed[6] that $X_t - \lambda^{-2}t \sim t^{2/3}$.

Here we prove:

Theorem 4. *Let $(X_t, t \geq 0)$ be the trajectory of a second class particle which is initially at the origin in the Hammersley's process with sinks, i.e. $\rho > 0$, and such that $\lambda\rho < 1$. Then one has, almost surely,*

$$\lim_{t\to\infty} \frac{X_t}{t} = Z$$

where Z is a random variable with the following distribution:

$$\mathbb{P}(Z \leq r) = \begin{cases} 0, & r \leq \rho^2, \\ \frac{\rho^{-1}-\sqrt{r^{-1}}}{\rho^{-1}-\lambda}, & \rho^2 < r \leq \lambda^{-2}, \\ 1, & \lambda^{-2} < r\,. \end{cases}$$

The almost sure convergence in the regime $\rho > 0$ and $0 \leq \lambda\rho < 1$ follows directly from Theorem 2, since the space-time path of a second-class particle can be regarded as a β-path. The description of the limit distribution[6] is obtained in Sec. 3.1.

In a further paper[7] we shall study these models in the regime $\lambda\rho > 1$ and we shall prove the almost sure convergence of an arbitrary β-path to the limit value ρ/λ. Differently from the regime $\lambda\rho \leq 1$, in this case the fluctuations of β-paths should be Gaussian. We note that, in the PNG context, this corresponds to the convergence of the competition interface to $(\rho - \lambda)/(\rho + \lambda)$. We also remark that, analogously to the second class particle in the totally asymmetric exclusion process,[9] the convergence to a deterministic limit value with Gaussian fluctuations is due to the development of a shock in the evolution of the macroscopic profile (hydrodynamic limit).

Overview. The paper is organized as follows. We begin by studying the stationary regime $\lambda\rho = 1$ (Sec. 2) and we prove Theorem 2 (in this regime) by using Remark 2 together with the concept of dual second-class particles. After that we relate the asymptotics for competition interfaces and second-class particles with the respective partial differential equations associated to the macroscopic evolution of the systems. In Sec. 3, we start by deriving the convergence in distribution of the second-class particle with coupling ideas of Ferrari and Kipnis[10] and general hydrodynamics results of Seppäläinen.[25] Next we use some results of Baik and Rains[3] concerning the tail of L_m, and the notion of δ-straightness of maximal

[6] We remark that the limit in distribution of the second-class particle when $\lambda\rho < 1$ was also identified by Cator and Dobrynin.[4]

paths introduced by Newman,[20] to prove the almost sure convergence of β-paths in the regime $0 \leq \lambda\rho < 1$ and to obtain the fluctuation upper bound.

2. STATIONARY GROWTH AND MACROSCOPIC DESCRIPTION

2.1. Dual Second-class Particle

The concept of a dual second-class particle was introduced by Cator and Groeneboom[5] to prove the convergence of the normal second-class particle in the stationary regime. Recall that to determine the process $t \to L(.,t)$ at point x we shift until time t the interval $[0, x]$ vertically through a realization and we follow the Hammersley interacting rule allowing particles to escape through the sinks. By symmetry, we can also introduce the dual process $x \to L^*(x, .)$ by running the same rule, but now from left to right, i.e. sinks for L become sources for L^* and sources for L become sinks for L^*. Notice that, in the stationary regime $\lambda\rho = 1$, both processes L and L^* have the same law. We denote X^* the second-class particle with respect to the dual process L^* and we denote by X_t^* the intersection between the space-time path of the dual second-class particle with $[0, \infty) \times \{t\}$. Trajectories of X and X^* are shown in Fig. 5.

Remark 3. The symmetry of the model and Remark 2 imply that if $\lambda\rho = 1$ then, almost surely,

$$\exists \lim_{t\to\infty} \frac{X_t^*}{t} = \frac{1}{\lambda^2}.$$

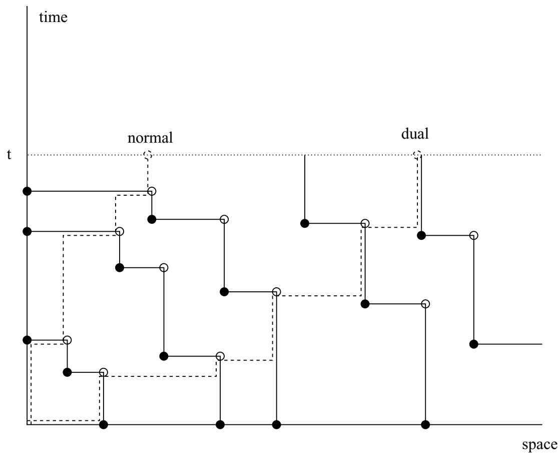

Fig. 5. Normal and dual second-class particles

An easy but useful observation is that the β-paths $(Q_n)_{\geq 0}$ and $(Q_n^*)_{\geq 0}$, which correspond to the normal and dual second-class particles are the left- and right-most β-paths (for all λ, ρ), respectively:

Lemma 1. *Let $(Q_n)_{\geq 0}$ and $(Q_n^*)_{\geq 0}$ be the β-paths that correspond to the normal and dual second-class particles respectively, and let $(P_n)_{\geq 0}$ be a β-path (recall we denote $P = (P(1), P(2))$. Then*

$$\frac{Q_n^*(2)}{Q_n^*(1)} \leq \frac{P_n(2)}{P_n(1)} \leq \frac{Q_n(2)}{Q_n(1)}.$$

Together with Lemma 1, Remark 3 implies the convergence of β-paths in the regime $\lambda\rho = 1$.

2.2. Macroscopic Evolution: Hamilton-Jacobi and Burges Equations

Clearly it is desirable to establish a correspondence between the microscopic structure of the interface generated by the collision between two PNG droplets and its macroscopic behavior. For the PNG droplet, it is known that if $\bar{h}(z, s)$ denotes the *macroscopic height profile* then $\bar{h}$ satisfies the Halmilton-Jacobi equation

$$\partial_s \bar{h} - v(\partial_z \bar{h}) = 0 \tag{2.2}$$

with the *inclination-dependent growth velocity* $v(u) = \sqrt{2 + u^2}$.[22,25] For the stationary growth $\rho\lambda = 1$ the solution is $\bar{h}(z, s) = sv(u) + zu$ with $u = (\rho - \lambda)/\sqrt{2}$. Since

$$v'(u) = \frac{u}{\sqrt{2 + u^2}} = \frac{\rho - \lambda}{\rho + \lambda} = \frac{1 - \lambda^2}{1 + \lambda^2}$$

we have that the line $\{z = v'(u)s\}$ is the macroscopic analogue of the competition interface. We also remark that, if one considers the fluctuations of the height profile then the slope $v'(u)$ plays an important rule: the height fluctuations are Gaussian with variance proportional to t except along the line $\{z = v'(u)s\}$ where they have the KPZ scaling form.[22,23]

In the Hammersley context, we have that if $u(x, t)$ denotes the *macroscopic density profile* then u satisfies the Burgers equation

$$\partial_t u + \partial_x g(u), \tag{2.3}$$

where $g(u) = 1/u$.[25] The *characteristics* $x(a, t)$ emanating from a are the solutions to the ordinary differential equation

$$x'(t) = g'(u(x, t))$$

with initial condition $x(0) = a$. In the stationary regime $\rho\lambda = 1$ the characteristics are given by the lines $x = t\lambda^{-2} + a$, which brings us the macroscopic analogue of the second-class particle, or more generally, of the β-paths.

We finish this section by saying a few words concerning the macroscopic evolution when $0 \le \lambda\rho < 1$. For the PNG droplet, the solution for the Hamilton-Jacobi equation is of the form $\bar{h}(cs, s) = sf(c)$ where f solves the equation $f(c) - cf'(c) = v(f'(c))$ and the parameter c is related to the local inclination $u = f'(c)$ by $c = -v'(u)$.[22,23] For instance, when $\lambda = \rho = 0$ we have the ellipsoidal shape $f(c) = \sqrt{2(1 - c^2)}$. With this information one obtains that the macroscopic height profile has a curved piece between the lines $\{z = s(\rho^2 - 1)(\rho^2 + 1)^{-1}\}$ and $\{z = s(1 - \lambda^2)(1 + \lambda^2)^{-1}\}$. In the Hammersley context, this corresponds to the development of a *rarefaction front* in the solutions of the Burgers equation, or equivalently, to the existence of infinitely many characteristics emanating from the origin. Theorem 2 shows that the macroscopic analogue of a β-path will be one of these characteristics.

3. RAREFACTION FRONT

3.1. Convergence in Distribution of Second Class Particles

The limit law of the second class particle follows from the computation below as well as from Cator and Dobrynin.[4] Let $\eta_t, t \ge 0$ be the point process obtained by starting with a Poisson Process of intensity λ in $(0, \infty)$ at time 0, and letting it develop according to Hammersley's process on $(0, \infty)$, with Poisson sinks of intensity ρ with $\lambda\rho < 1$ and a Poisson point process of intensity 1 in the interior of the first quadrant. Furthermore, let $\eta_t^h, t \ge 0$ be the process coupled to $\eta_t, t \ge 0$, by using the same points in the first quadrant and on the t-axis as used for η. At time 0, we consider the same sources on the interval (h, ∞) and on the interval $[0, h]$ we add an independent Poisson process of intensity $\rho^{-1} - \lambda$. Denote by $\eta_t[x, y]$ the number of particles in the interval $[x, y]$ at time t and similarly by $\eta_t^h[x, y]$ for the coupled process.

Let

$$F_h^\eta(r, t) = \eta_t[0, r] - \eta_t^h[0, r]$$

and

$$F_{h,\epsilon}^\eta(r, t) = F_h^\eta(r\epsilon^{-1}, t\epsilon^{-1}) .$$

Notice that in the absence of extra sources in $[0, h]$ we have $F_{h,\epsilon}^\eta(r, t) = 0$. If there is a unique source in $[0, h]$ coming from the Poisson point process of intensity $\rho^{-1} - \lambda$ (which happens with probability $(\rho^{-1} - \lambda)he^{-\rho^{-1}h}$) we have a discrepancy which behaves like a second class particle. Denoting it's position at time t by $X_{t,h}$ we get that $F^{\eta}_{h,\epsilon}(r,t) = -1$ iff $X_{t\epsilon^{-1},h} \le r\epsilon^{-1}$. Therefore

$$\mathbb{E}\left(F^{\eta}_{h,\epsilon}(r,t)\right) = -(\rho^{-1}-\lambda)he^{-\rho^{-1}h}\mathbb{P}(X_{t\epsilon^{-1},h} \le r\epsilon^{-1}) + o(h)$$

Dividing by h and taking limit when h and ϵ go to 0 we get

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathbb{E}\left(\frac{F^{\eta}_{h,\epsilon}(r,t)}{h}\right) = -(\rho^{-1}-\lambda)\lim_{\epsilon\to 0}\mathbb{P}(X_{t\epsilon^{-1}} \le r\epsilon^{-1})$$

On the other hand, by combining the stationarity of the process η^h on $[0,h]\times[0,t]$ with the fact that the number of particles at time t on $[0,r]$ equals the number of space-time curves crossing the rectangle $[h,r]\times[0,t]$ plus the number of sources on $[0,h]$, we get

$$\begin{aligned}\mathbb{E}\left(F^{\eta}_{h,\epsilon}(r,t)\right) &= \mathbb{E}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h, r\epsilon^{-1}] - \eta^h_0[0,h]\right)\\ &= \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h, r\epsilon^{-1}] = 1, \eta^h_0[0,h] = 0\right)\\ &\quad - \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h, r\epsilon^{-1}] = 0, \eta^h_0[0,h] = 1\right) + o(h).\end{aligned}$$

Since

$$\begin{aligned}&\mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h, r\epsilon^{-1}] = 1, \eta^h_0[0,h] = 0\right)\\ &\quad - \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h, r\epsilon^{-1}] = 0, \eta^h_0[0,h] = 1\right)\\ &= \mathbb{P}(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h, r\epsilon^{-1}] = 1) - \mathbb{P}\left(\eta^h_0[0,h] = 1\right) + o(h),\end{aligned}$$

and

$$\mathbb{P}\left(\eta^h_0[0,h] = 1\right) = (\rho^{-1}h)e^{-\rho^{-1}h},$$

together with the hydrodynamics results of Seppäläinen[25], this yields

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathbb{E}\left(\frac{F^{\eta}_{h,\epsilon}(r,t)}{h}\right) = u(r,t) - \rho^{-1}$$

where $u(r,t)$ is the unique entropic solution of (2.3), given by

$$u(x,t) = \rho^{-1}\,1\{\rho^{-2}x \le t\} + \sqrt{tx^{-1}}\,1\{\lambda^2 x \le t < \rho^{-2}x\} + \lambda\,1\{t < \lambda^2 x\}.$$

Consequently,

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\epsilon X_{t\epsilon^{-1}} \le r\right) = \frac{\rho^{-1}-u(r,t)}{\rho^{-1}-\lambda}$$

which gives the limit law of the second-class particle.

Concerning the limit angle θ of the competition interface when $0 \le \lambda\rho < 1$, at the present moment we do not know how to calculate its distribution but we do know it must be random. In fact, this is true for any β-path and this is a consequence of the fact that every β-path is always in between the trajectories of the normal and the dual second-class particles. Indeed, if there exists $a \in [\lambda^2, \rho^{-2}]$ such that with probability one $\tan(\theta) = a$ then with probability one $Z \in [\lambda^2, a]$ and $Z' \in [a, 1/\rho^2]$, where Z and Z' denote the limit value of the normal and the dual second-class particles. However, this leads to a contradiction since we do know that Z and Z' have continuous distributions with support on $[\lambda^2, \rho^{-2}]$.

3.2. δ-straightness of Maximal Paths

In[3] Baik and Rains used an analytical point of view to study the asymptotics of L_m (see also Ref. 2 for the regime $\lambda = \rho = 0$). The following is a consequence of their bounds for the tail of L_m: if $\lambda_1, \lambda_2 \in [0, 1)$ then there exists constants $c_j > 0$ such that for $k \le -c_1$ and $t \ge c_2$ we have

$$\mathbb{P}\big(L_{\lambda_1}^{\lambda_2}(t) - 2t \le kt^{1/3}\big) \le c_3 e^{-|k|^3}, \tag{3.4}$$

and that for $k \ge c_1$ and $t \ge c_2$ we have

$$\mathbb{P}\big(L_{\lambda_1}^{\lambda_2}(t) - 2t \ge kt^{1/3}\big) \le c_4 e^{-c_5 k^{3/2}}, \tag{3.5}$$

where $L_{\lambda_1}^{\lambda_2}(t) := L_m(t, t)$, λ_1 is the intensity of points in x-axis and λ_2 is the intensity of points in the t-axis (see Eqs. (5.2) and (5.14) in Ref. 3). Notice that for $\lambda\rho < 1$ and $\rho^2 < x/t < 1/\lambda^2$,

$$\mathbb{P}\big(L_{\lambda}^{\rho}(x, t) - 2\sqrt{xt} > k\big(\sqrt{xt}\big)^{1/3}\big) = \mathbb{P}\big(L_{\lambda_1}^{\lambda_2}(\sqrt{xt}) - 2\sqrt{xt} > k(\sqrt{xt})^{1/3}\big)$$

where $\lambda_1 = \lambda\sqrt{x/t}$, $\lambda_2 = \rho\sqrt{t/x} \in [0, 1)$. Therefore, if we denote $\alpha(x, t) := 2\sqrt{xt}$, for all $\epsilon > 0$ one can find constants $b_j > 0$ such that if $\rho^2 + \epsilon < x/t < 1/\lambda^2 - \epsilon$ then

$$\mathbb{P}\big(|L_m(0, (x, t)) - \alpha(x, t)| > k(\sqrt{xt})^{1/3}\big) \le b_1 e^{-b_2 k^{3/2}}. \tag{3.6}$$

Since the work of Kardar, Parisi and Zhang[17] it is known there is a strong relation between the fluctuations of $L_m(0, Q)$ and the deviations of a maximal path $\gamma(0, Q)$, connecting 0 to Q, about the line segment $[0, Q]$. It is expected that the scaling relation $\chi = 2\xi - 1$ holds, if

$$|L_m(0, Q) - \alpha(Q)| \sim |Q|^{\chi} \text{ and } \sup_{P \in \gamma(0, Q)} d(P, [0, Q]) \sim |Q|^{\xi},$$

where $d(A, B)$ denotes the euclidean distance between A and B. Since for this Poisson last-passage model we do know that $\chi = 1/3$, we should have that $\xi = 2/3$. Johansson[16] proved that $\xi = 2/3$ when $\lambda = \rho = 0$ by using a geometric idea developed by Newman,[20] which is based on the curvature properties of the limit shape of the rescaled growth process $n^{-1}G_n$. Newman also introduced the notion of δ-straightness of a maximal path as follows. For each Poissonian point

P, let $R^{\text{out}}(P)$ be the set of all Poissonian points Q, $P \prec Q$, such that there is a maximal path from 0 to Q passing through P. For $\theta \in (0, \pi/4)$ denote $Co(P, \theta)$ the cone with axis through P and 0 and with angle θ. Let $\delta > 0$. We say that $R^{\text{out}}(P)$ is δ-straight if for some constant $c > 0$

$$R^{\text{out}}(P) \subseteq Co(P, c|P|^{-\delta}).$$

Proposition 1. *Assume that* $0 \leq \lambda\rho < 1$. *For any* $\epsilon > 0$ *and* $\delta \in (0, 1/3)$*, almost surely, for all but finitely many Poissonian points* $P = (P(1), P(2))$ *with* $P(2)/P(1) \in [\lambda^2 + \epsilon, \rho^{-2} - \epsilon]$ *one has that* $R^{\text{out}}(P)$ *is* δ*-straight.*

For $\lambda = \rho = 0$ this is exactly Lemma 2.4 of Wüthrich.[26] To avoid repetitions we give just a sketch of the proof which repeats the geometric argument of Newman.

Proof of Proposition 1. Denote by A_P the set of Poisson points Q that satisfies: (i) $P \prec Q$; (ii) $ang(P, Q) \in [\alpha(P)^{-\delta}, 2\alpha(P)^{-\delta}]$; (iii) $\alpha(Q) \leq 2\alpha(P)$. Notice that $\alpha(aP) = a\alpha(P)$ and so $\alpha(P)$ has the same order of $|P|$. If $|P|$ is sufficiently large then we must have that for all $Q \in A_P$, $Q(2)/Q(1) \in [\lambda^2 + \epsilon/2, \rho^{-2} - \epsilon/2]$. Now, assume there is $Q \in R^{\text{out}}(P) \cap A_P$. Then P belongs to some maximal path from 0 to Q which implies that

$$L_m(0, Q) = L_m(0, P) + L(P, Q),$$

and so

$$\begin{aligned}&(\alpha(Q) - L(0, Q)) + (L(0, P) - \alpha(P)) + (L(P, Q) - \alpha(Q - P)) \\ &= \alpha(Q) - \alpha(P) - \alpha(Q - P) =: \Delta(P, Q). \qquad (3.7)\end{aligned}$$

By Lemma 2.1 of Wüthrich[26] (which is the desired curvature property for α), for such a P and Q,

$$\Delta(P, Q) \geq |P|^{1-2\delta}.$$

By using (3.6) one can prove that if $\delta \in (0, 1/3)$, or equivalently $(1 - 2\delta) = \chi \in (1/3, 1)$, then (3.7) does not occur for all but finitely many P.

As a consequence of the preceding paragraph, one gets that for all but finitely many P, if $Q \in R^{\text{out}}(P)$ and $\alpha(Q) \leq 2\alpha(P)$ then either

$$ang(P, Q) \leq \alpha(P)^{-\delta}$$

or

$$ang(P, Q) > 2\alpha(P)^{-\delta}.$$

Since, for sufficiently large $|P|$, to go from $\partial Co(P, \alpha(P)^{-\delta})$ to some point $Q \in Co(P, 2\alpha(P)^{-\delta})^c$ a maximal path must pick one Poissonian point Q' with

$$ang(P, Q') \in [\alpha(P)^{-\delta}, 2\alpha(P)^{-\delta}],$$

the second item in the above two possibilities can be deleted.

Therefore, for all but finitely many P if $Q \in R^{\text{out}}(P)$ and $\alpha(Q) \le 2\alpha(P)$ then

$$ang(P, Q) \le \alpha(P)^{-\delta}.$$

Now we claim that this implies δ-straightness. In fact, for every $Q \in Co(P, \epsilon_1)$ the cone $Co(Q, \epsilon_2)$ is contained in the cone $Co(P, \epsilon_1 + \epsilon_2)$. By induction, for

$$\epsilon_m(P) = \sum_{j=0}^{m-1} (2^j \alpha(P))^{-\delta},$$

$$R^{\text{out}}(P) \subseteq Co(P, \epsilon_m(P)) \bigcup_{\alpha(Q) \ge 2^m \alpha(P)} R^{\text{out}}(Q).$$

By noticing that $\epsilon_m(P) \le c$, for some constant $c = c(\delta) > 0$, one can easily finish this proof. $\square$

As a consequence of the δ-straightness property of maximal paths we have:

Corollary 1. *Let $a, a' \in (\rho^2, \lambda^{-2})$ with $a < a'$. Almost surely, if $(Q_i)_{i \ge 1}$ and $(Q'_j)_{j \ge 1}$ are two sequences of Poissonian points such that $Q_i \prec Q_{i+1}, Q'_j \prec Q'_{j+1}$, $\lim_{i\to\infty} Q_i = \lim_{j\to\infty} Q'_j = \infty$ and*

$$\limsup \frac{Q'_j(2)}{Q'_j(1)} < 1/a' < 1/a < \liminf \frac{Q_i(2)}{Q_i(1)}$$

then there are only finitely many i such that, for some j, $Q'_j \in R^{\text{out}}(Q_i)$. Analogously, there are only finitely many j such that, for some i, $Q_i \in R^{\text{out}}(Q'_j)$.

Proof of Corollary 1. Divide the positive quadrant into 5 regions as follows:

$$C_0 := \{0 \le t \le \lambda^2 x\},$$

$$C_1 := \{0 \le \lambda^2 x \le t \le x/a\},$$

$$C_2 := \{0 \le x/a \le t \le x/a'\},$$

$$C_3 := \{0 \le x/a' \le t \le \rho^2\},$$

and finally,

$$C_4 := \{0 \le x/\rho^2 \le t\}.$$

Pick a $\delta \in (0, 1/3)$ and notice that, almost surely, for sufficiently large $|Q|$:

1. $R^{\text{out}}(Q)$ is δ-straight;
2. If $Q \in C_0$ and $Q \prec Q' \in C_3$ then every optimal path from Q to Q' has a Poissonian point in C_1, and if $Q \in C_4$ and $Q \prec Q' \in C_1$ then every optimal path from Q to Q' has a Poissonian point in C_3;
3. If $Q \in C_1$ then $Co(Q, c|Q|^{-\delta}) \cap (C_3 \cup C_4) = \emptyset$ and if $Q \in C_3$ then $Co(Q, c|Q|^{-\delta}) \cap (C_1 \cup C_0) = \emptyset$.

Now, assume that $Q'_j \in R^{\text{out}}(Q_i)$, $|Q'_j|, |Q_i| \geq M$ and $Q'_j(2)/Q'_j(1) < 1/a'$. If $Q_i \in C_3$ then, by (1) and (3), $Q'_j \notin (C_1 \cup C_0)$, which yields to a contradiction. If $Q_i \in C_4$, by (2), there exists a $\bar{Q}_i \in C_3$ such that $Q'_j \in R^{\text{out}}(\bar{Q}_i)$, and so, by (1) and (3), we also get a contradiction. Since, by assumption, $Q_i \in C_3 \cup C_4$ for all but finitely many i, there are only finitely many i such that, for some j, an optimal path from 0 to Q'_j passes through Q_i. The same proof works for the analogous case.

3.3. Asymptotics for β-paths

The idea to control the deviations of a β-paths, when $0 \leq \lambda\rho < 1$, is to show that if $(P_n)_{n\geq 0}$ is a β-path then for all $n \geq 1$ we can construct two maximal paths, both starting from $(0, 0)$ and ending at P_n, such that the path $(P_0, \ldots, P_n)$ is enclosed by them (see Fig. 6).

Lemma 2. *Almost surely, if $(P_n)_{n\geq 0}$ is a β-path then for all $n \geq 0$ there exist two maximal paths γ_n^+ and γ_n^- in $\Gamma(0, P_n)$ such that γ_n^+ is above $(P_0, \ldots, P_n)$ and γ_n^- is below $(P_0, \ldots, P_n)$.*

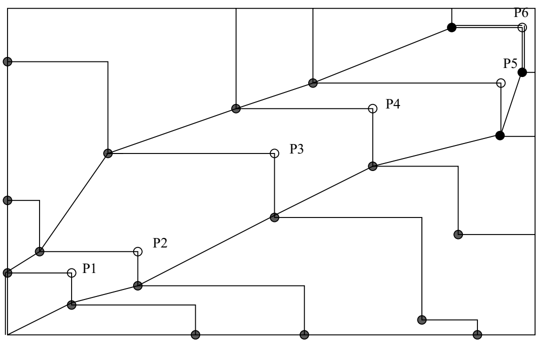

Fig. 6. Two geodesics enclosing a beta-path.

Proof of Lemma 2. Let $G_n^+ = (G_n^+(1), P_n(2))$ be the Poissonian point (coming from one of the three Poisson point processes) that first appears to the left (hand-side) of P_n in level ∂G_n. Fix $Q \in \mathbb{R}_+^2$ and let $A_Q = \{P \prec Q\}$. Suppose that $G_n^+, \dots, G_{n-k}^+$ have already been defined for $k < n$. Then set $G_{n-(k+1)}^+$ to be the first Poissonian point in level $\partial G_{n-(k+1)} \cap A_{G_{n-k}^+}$ to the left of $P_{n-(k+1)}$. Notice that if one of the G_k^+ belongs to the t-axis then $G_0^+, \dots, G_{k-1}^+$ belong to the t-axis as well. By construction, the oriented path $(G_0^+, \dots, G_n^+, P_n)$ is a geodesic (since it picks one point in each level behind P_n) which is always above $(P_0, \dots, P_n)$. Similarly, we can construct a geodesic $(G_0^-, \dots, G_n^-, P_n)$ which is below $(P_1, \dots, P_n)$. In this case, we proceed as follows: let $G_n^- = (P_n(1), G^-(2)_n)$ be the Poissonian point that first appear to the right of P_n in level ∂G_n. Suppose that $G_n^-, \dots, G_{n-k}^-$ have already been defined for $k < n$. Then set $G_{n-(k+1)}^-$ to be the first Poissonian point in level $\partial G_{n-(k+1)} \cap A_{G_{n-k}^-}$ to the right of $P_{n-(k+1)}$. Notice that if one of the G_k^- belongs to the x-axis then $G_0^-, \dots, G_{k-1}^-$ belong too. By construction, the path $(G_0^-, \dots, G_n^-, P_n)$ is a geodesic which is always below $(P_0, \dots, P_n)$. □

Proof of Theorem 2 (when $0 \leq \lambda\rho < 1$**).** First we claim that, almost surely, if $(P_n)_{n\geq 1}$ is a β-path then

$$\lambda^2 \leq \liminf_{n\to\infty} \frac{P_n(2)}{P_n(1)} \leq \limsup_{n\to\infty} \frac{P_n(2)}{P_n(1)} \leq \frac{1}{\rho^2} . \tag{3.8}$$

By Lemma 1, to obtain (3.9) it suffices to show

$$\lambda^2 \leq \liminf_{n\to\infty} \frac{Q_n^*(2)}{Q_n^*(1)} \leq \limsup_{n\to\infty} \frac{Q_n(2)}{Q_n(1)} \leq \frac{1}{\rho^2} .$$

where $(Q_n)_{\geq 1}$ and $(Q_n^*)_{\geq 1}$ are the β-paths corresponding to the normal and dual second-class particles, respectively.

The second inequality follows by coupling the Hammersley's process with parameters λ, ρ, with the stationary Hammersley's process with parameters $1/\rho, \rho$. Since $1/\rho > \lambda$ (more sources for the stationary process), $X_t(\lambda, \rho)$ moves to the right faster than $X_t(1/\rho, \rho)$, i.e. the normal second-class particle for the original process is always to the right of the normal second-class particle for the stationary process.[5] Together with Remark 3, this yields the second inequality. To show the first inequality, we couple the Hammersley process, with parameters λ, ρ, with the stationary process with parameters $\lambda, 1/\lambda$ (more sinks for the stationary process) and repeat the same argument for the dual second-class particle.

By (3.8), if $(P_n)_{n\geq 1}$ does not converge then there exist $b < a < a' < b'$ such that

$$\lambda^2 \leq \liminf_{n\to\infty} \frac{P_n(2)}{P_n(1)} < \frac{1}{b'} < \frac{1}{a'} < \frac{1}{a} < \frac{1}{b} < \limsup_{n\to\infty} \frac{P_n(2)}{P_n(1)} \leq \frac{1}{\rho^2} . \tag{3.9}$$

Now let $m < n$ and assume that

$$\frac{P_m(2)}{P_m(1)} < \frac{1}{b'} < \frac{1}{b} < \frac{P_n(2)}{P_n(1)} .$$

Consider the optimal path γ_n^-, giving by Lemma 2, which connects 0 to P_n. Since γ_n^- lies below $(P_0, \ldots, P_m, \ldots, P_n)$, if $|P_m|$ is sufficiently large then one can find $Q', Q \in \gamma_n^-$ (Poissonian points) with $Q \in R^{\text{out}}(Q')$ and such that

$$\frac{Q'(2)}{Q'(1)} < \frac{1}{a'} < \frac{1}{a} < \frac{Q(2)}{Q(1)} .$$

Therefore, if (3.9) occurs then one can construct two sequences of Poissonian points, say $(Q'_j)_{j\geq 1}$ and $(Q_i)_{i\geq 1}$, with $Q_i \in R^{\text{out}}(Q'_j)$ and such that

$$\frac{Q'_j(2)}{Q'_j(1)} < \frac{1}{a'} < \frac{1}{a} < \frac{Q_i(2)}{Q_i(1)}$$

for all $i, j \geq 1$. By Proposition 1, this occurs with probability 0 and thus $(P_n)_{n\geq 1}$ must converge almost surely. □

Proof of Theorem 3. From Theorem 2, we have the almost sure convergence of the normal and dual second-class particles and, by a previous calculation (Sec. 3.1), their limits have a continuous distribution. Combining this with Lemma 1, one gets that, almost surely, there exists a sufficiently small (random) $\epsilon > 0$ such that for all β-paths $(P_n)_{n\geq 0}$, and n sufficiently large,

$$\lambda^2 + 2\epsilon < \frac{P_n(2)}{P_n(1)} < \rho^{-2} - 2\epsilon .$$

Choose a sufficiently large M such that if $|P| \geq M$ and

$$\lambda^2 + 2\epsilon < \frac{P(2)}{P(1)} < \rho^{-2} - 2\epsilon$$

then for any $Q \in Co(P, c|P|^{-\delta})$,

$$\frac{Q(2)}{Q(1)} \in [\lambda^2 + \epsilon, \rho^{-2} - \epsilon] .$$

Denote by θ_m the angle in $[0, \pi/2]$ such that $\tan(\theta_m) = \frac{P_m(2)}{P_m(1)}$ and such that $|P_m| \geq M$ and that

$$\frac{P_n(2)}{P_n(1)} > \tan(\theta_m + 3c|P_m|^{-\delta})$$

for some $n \geq m$. Consider the the maximal path γ_n^- giving by Lemma 2. Since γ_n^- lies below $(P_0, \ldots, P_m, \ldots, P_n)$, for sufficiently large M, there exist $Q, Q' \in \gamma_n^-$

with $Q' \in R^{\text{out}}(Q)$ and such that

$$\frac{Q(2)}{Q(1)} < \tan(\theta_m + c|P_m|^{-\delta}) \text{ and } \frac{Q'(2)}{Q'(1)} > \tan(\theta_m + 2c|P_m|^{-\delta}) .$$

Since $|Q| \sim |P_m|$, this would imply that $R^{\text{out}}(Q)$ is not δ-straight which, by Proposition 1, occurs with probability 0. If

$$\frac{P_n(2)}{P_n(1)} < \tan(\theta_m - 3c|P_m|^{-\delta})$$

one can repeat the same argument, but now considering the maximal path γ_n^+ that lies above $(P_0, \ldots, P_m, \ldots, P_n)$, to prove that it does not happen with probability 1. Therefore, for sufficiently large m and for all $n \geq m$,

$$ang(P_m, P_n) \leq 3c|P_m|^{-\delta} .$$

By sending $n \to \infty$, one gets Theorem 3. $\square$

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